3901. (a) Find t when the particle is at the origin, and substitute this into expressions for $\dot{x}$ and $\dot{y}$.

(b) Set $y = 0$ and solve (take out a factor of $(t-2)$. You want the first positive t value. Find the gradient at this point, and use $\tan\theta = m$.

3902. Assume, for a contradiction, that $\sqrt{2} - \sqrt{3} = \frac{p}{q}$, where p and q are integers. Square both sides.

3903. (a) Once you've understood what the question means, the answer is fairly straightforward. Try replacing 100 by 10, and 20 by 2. Then you can list the relevant integers $\{1, 2, ..., 10\}$ and work visually.

(b) Consider the lowest possible value of a and work out b using your result from (a). Then consider the highest possible value of b and find a likewise. This will give you minimum and maximum values of $a + b$.

3904. Write the denominator as $\left(4 + qx^2\right)^{-3}$, and take out a factor of 4^{-3}. Then perform the expansion and equate coefficients.

3905. (a) Multiply out and factorise.

(b) There are three factors in the result from part (a). Including or excluding each of these in a product produces $2^3 = 8$ factors. The task is then to guarantee that they are all distinct.

3906. A queen threatens a different number of squares, depending on her location. So, condition by the location of the first queen.

3907. Find the equation of the tangent, and set up a quartic equation for intersections. Take out $(x-1)^2$, which you know must be present as $x = 1$ is a point of tangency. Show that the remaining quadratic factor has no real roots.

3908. You need to construct an explicit irrational given any two distinct rationals $a < b$. There are many ways of doing this. One way is to find an irrational number smaller than $b - a$, and add it to a.

3909. (a) Differentiate implicitly with respect to x. Then multiply both sides by $2\sqrt{x-y}\sqrt{x+y}$.

(b) Substitute $\frac{dy}{dx} = 1$ without rearranging further, and solve.

3910. (a) Draw a clear sketch, and consider the line of symmetry in which C_2 is a reflection of C_1.

(b) Find the area enclosed by C_1 and $y = \sqrt{3}x$, and double it.

3911. The key fact is that the forces in the interaction are a Newton III pair, i.e. as 1D vectors they are F and $-F$. Find acceleration for each particle, and then use $v - u = at$. Eliminate Ft from the equations and rearrange to "total momentum before equals total momentum after."

3912. Find the derivative of y by the product rule, and substitute in.

3913. (a) Divide top and bottom by x^2 before taking the limit.

(b) Factorise top and bottom, looking to cancel a factor of $(10x - 21)$.

3914. Set up inequalities requiring the first derivative to be positive and also the second derivative to be positive. Solve these simultaneously.

3915. Firstly, use $\sin x \approx x$. Then convert to the form $(1+\mathrm{f}(x))^{-1}$, where $\mathrm{f}(x)$ is a quadratic in x. Use the generalised binomial expansion, neglecting terms in x^3 and higher.

3916. (a) Differentiate with respect to y and set $\frac{dx}{dy} = 0$. Substitute the resulting equation back into that of the curve.

(b) Make x the subject of the equation of the curve.

(c) The equation $y - 1 = m(x - 2)$ describes all straight lines through P. Work visually: you don't need to do any calculation here.

3917. Let $z = \sin x$ and $y = \tan x$. Find $\frac{dz}{dy}$ using the chain rule. Setting this to zero for SPs, you'll find that the resulting equation does have real roots: show that they are not in the domain of $\tan x$.

3918. Solve the quartic using a polynomial solver. Then find the probability that the first root chosen is positive and the second negative. Multiply this by the number of orders in which $+-$ can occur.

3919. Check Δ to show that the equation has no real roots. Then plug in $z = 4 \pm 2i$, simplifying the square i^2 to -1.

3920. (a) You can establish the behaviour as $x \to -\infty$ immediately, since $e^x \to 0$. For behaviour as $x \to \infty$, divide top and bottom by e^x.

(b) Set the derivative to zero for SPs.

(c) Substitute in $x = -a$ to the RHS. Multiply top and bottom by e^a.

(d) Use the previous parts of the question.

3921. Put the fractions over a common denominator, and use the first Pythagorean trig identity.

3922. This is a cubic in $\ln x$. Multiply by $\ln x$ and use a polynomial solver.

3923. Only the first statement is true. Come up with a counterexample to the second. It's easier to sketch one rather than describe it algebraically.

3924. (a) Multiply up by the denominators and gather like terms.
(b) Solve the equation in (a) as a quadratic in x. Take out a factor of y from the result, and you should have two roots $x = ay$ and $x = by$. Show that both are well defined, with $a \neq b$.

3925. (a) The table is a 3×3 grid. On one axis is the card A gives B, on the other is the card B gives A.
(b) Restrict the possibility space from the original nine outcomes to those in which B ends up with at least a pair, then use $p = \frac{\text{successful}}{\text{total}}$.

3926. In each case, factorise fully and consider the nature and multiplicity of the roots.

3927. Square both equations and reciprocate the second. Then combine the two using the third Pythagorean trig identity.

3928. Draw a clear sketch. Consider the transformation as reflection in the line $x + y = 0$ followed by a translation. Find the equation of the reflection of $y = x^2$ in the line $x + y = 0$. Then, by finding the vertex of the transformed graph (in terms of k), translate the curve appropriately.

3929. (a) Consider the first Pythagorean trig identity, or equivalently the equation of a unit (semi)circle.
(b) Write $\theta = \pi - \arccos\frac{11}{14} - \arccos\frac{13}{14}$. Then use the identity $\sin(\pi - x) \equiv \sin x$ to get rid of the π. Then expand with a compound-angle formula, and simplify using part (a).

3930. Find the sum of all the integers from 1 to $3k$. Also find the sum of the integers from 1 to $3k$ which are divisible by 3. Both of these are the sums of APs. Subtract the one from the other.

3931. (a) The distribution of X is binomial.
(b) The expectation in $np = 1$. So, find simplified expressions for $\mathbb{P}(X = 0)$ and $\mathbb{P}(X = 1)$. Use these to find $\mathbb{P}(X = 1 \mid X \leq 1)$, simplifying your answer.

3932. Consider the boundary case, in which the circle is the biggest it can be. Its centre must be on $y = x$, so the points of tangency must be points at which the gradients of the curves are 1.

3933. Substitute the latter into the former. Rearrange to make the square root the subject, and square both sides. Note that, in doing so, you may (you do, in fact) introduce new solution points. Solve simultaneously, and check any proposed solution points rigorously in both equations.

3934. Each point on each of these graphs lies on $y = \sin x$ or $y = -\sin x$. The question is about identifying which bits of the curves are introduced or removed by adding modulus signs.

3935. (a) Use the quotient rule.
(b) Integrate the result from (a).

3936. (a) Since $a, b > 0$, the endpoints of the range are each reciprocated (and their order is reversed).
(b) Write the numerator as $f(x) + 1 - 1$ and split the fraction up.

3937. Draw a clear sketch of the scenario. There are two possible normals at $x = k$, but, by symmetry, they must cross the x axis at the same place. Choose one wlog. Find its equation, with coefficients in terms of k. Find the x intercept.

3938. (a) Add the accelerations.
(b) Use normal projectile methods, with the value of acceleration from part (a).
(c) Repeat (b), with acceleration g ms^{-2}.

3939. The derivative of a^x is $\ln a \cdot a^x$. Use this to find stationary points. Find the intercepts and consider the behaviours as $x \to \pm\infty$.

3940. Draw a sketch, labelling A, B, C, D in order. Then use the cosine rule in $\triangle ABC$ and $\triangle ABD$, with angles ABC and DAB. The second equation will have elements that don't appear in the required result. Replace these, using the fact that $ABCD$ is a parallelogram. You'll need to use

$$\cos(180° - \theta) \equiv -\cos\theta.$$

Finally, add the two equations.

3941. There are two different boundary cases here, in which friction is limiting at either of the surfaces. This question concerns the boundary case in which friction *between* the blocks is limiting. Beyond this limit (very large m_3), the lower block will slip out from underneath the upper block. So, consider this boundary case.

3942. The implication is only forwards. Consider the constant of integration for a counterexample.

3943. Take out a factor of x from the fourth roots, then expand them binomially. You only need the first two terms in each case. Those without factors of h should cancel. This allows division of top and bottom by h, after which you can safely take the limit.

3944. For success, one couple must sit opposite each other. Start by supposing this is A_1, A_2. Show that there are 48 ways in which this can happen. Generalise to consider other couples sitting apart, to count 144 successful outcomes.

— Alternative Method —

For a conditioning method, suppose that A_1 and A_2 are to be the couple sitting apart. Place A_1 wlog. Find the probability that A_2 sits opposite. Place B_1 wlog. Then run through the others. At the end, there's a factor to multiply by, because it doesn't have to be A_1 and A_2 who sit apart.

3945. (a) Count!

(b) Consider the following fact: a visit to any land mass, except the start and endpoints of the walk, must use <u>two</u> bridges that connect it.

3946. (a) Write the numerator as $\frac{a}{c}(cx+d)+k$, for some k in terms of a, b, c, d.

(b) There is a vertical asymptote, where the denominator is zero, and also a horizontal asymptote, to be found by considering $x \to \infty$. The graph is then a stretched translation of the standard reciprocal graph $y = \frac{1}{x}$.

3947. This is true of a polynomial function, but not of e.g. a reciprocal function.

3948. (a) Quote the standard formula.

(b) Plug the numbers in.

(c) Draw a sketch of $y = \frac{1}{x} - p$, identifying the root. Then consider the iteration graphically: if the tangent line crosses the x axis to the left of the origin, then the iteration will diverge to negative infinity. Consider the boundary case, in which the tangent line passes through the origin.

3949. Eliminate z from the first and third equations, to produce two equations in x and y.

3950. Consider a specific value $x = \alpha$.

3951. (a) Use the quadratic formula, showing that only the positive root is valid.

(b) Consider the fact that 4 becomes negligible compared to x^6 as x gets large.

(c) Use (b) to show that the curve is convex on $(\approx 1, \infty)$, and then consider the other point of inflection.

(d) The last piece of the puzzle is the behaviour as $x \to -\infty$. Find this, then join the dots.

3952. You can use the inclusion-exclusion principle here, if you know it, or else proceed visually on a Venn diagram in a way that follows the same logic.

Call the three-way intersection probability p. Then work outwards from the middle of the Venn diagram, writing the three two-way intersections in terms of p, and then the remaining regions. Then equate the sum of all the probabilities to 1 and solve.

3953. Differentiate by the chain rule, then factorise fully. For one factor, use $\tan x \equiv \frac{\sin x}{\cos x}$.

3954. Draw a very clear force diagram. Let $\angle BAP = 2\theta$. Express both $\sin\theta$ and $\cos\theta$ in terms of l. Assume friction is at $F_{\text{max}} = \mu R$, and resolve horizontally and vertically. Eliminate R and rearrange to make μ the subject.

3955. Complete the square or use calculus to find the vertex. Calculate the position of the transformed vertex. Draw a sketch to work out the equation of the new graph.

3956. Rearrange to $\text{f}(x) = 0$. Then take out a factor of x. Spot a simple root, and take out the corresponding factor. This leaves a quadratic.

3957. Remember the constant of integration!

3958. ① Set the first derivative to zero for SPs. Use the quadratic formula and show, using $b^2 - 3ac$, that there are two SPs. The formula gives an easy way to find their mean.

② Show that the second derivative has a single factor of $(3ax + b)$.

3959. (a) Equate the x coordinates, and solve for time t. Then verify that the y coordinates are also the same at this time.

(b) Find the distance between the boats at $t = 0$ and halve it.

3960. (a) Set up the cosine rule, in the form $\cos B = ...$ Then differentiate implicitly with respect to t. Note that, since the rates of change of a and c are instantaneously zero, they can be treated as constants.

(b) The triangle is instantaneously right-angled. Use this to calculate $\sin B$, and substitute in to the result from (a).

3961. (a) Show that $x_1 = a$ and $y_1 = 0$. Use these values to calculate x_2 in terms of a.

(b) Start again and work in a similar way to find y_2, x_3 and y_3. Solve the resulting quartic in b by factorising, or by using a polynomial solver.

3962. (a) Show that the normal gradient is $\frac{-1}{3p^2}$. Then use $y - y_1 = m(x - x_1)$ and rearrange.

(b) The shortest path to the cubic lies along the normal. So, substitute the point $(14, 7)$ into the equation from (a). This will give you a quintic in p. Since it's a quintic, you can't use a polynomial solver. Set up the Newton-Raphson iteration and solve to show $p = 2$. This will give you $(2, 8)$ as the point on $y = x^3$ closest to $(14, 7)$. Show that the distance is $\sqrt{145}$. Then repeat the procedure for $(13, -3)$ and compare.

3963. (a) Resolve the initial velocity into $v_x = u \cos\theta$ and $v_y = u \sin\theta$, and find displacements.

(b) Set $y = 0$ and find the time of flight. Then substitute this into your formula for x.

(c) Use $\sin 2\theta \equiv 2 \sin\theta \cos\theta$.

3964. (a) Equate coefficients, using ${}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$.

(b) Divide the first equation by the second, and simplify. Use the fact that a is an integer.

(c) Use the fact that $b - 1$ is even to show that b must contain a factor of 5. Then use the fact that b is a single-digit integer.

3965. Set y to zero for the x intercept. Then make y the subject and set up an integral. Multiply it out and integrate, treating c as a constant throughout.

3966. (a) Consider the root of the denominator.

(b) Either write the numerator as $(2x - 1)\,\mathrm{f}(x)$, where $\mathrm{f}(x)$ is a polynomial, or use polynomial long division.

(c) Consider the graph as a cubic plus a reciprocal term. No calculation is needed here.

3967. Use the fact that

$$\log_p q \equiv \tfrac{1}{\log_q p}.$$

Start with the RHS and simplify, using one other log rule on the way.

3968. Consider the fact that the range of both $\sin t$ and $\sin kt$ is $[-1, 1]$. This allows you to find potential values of t without solving any major equations.

3969. Square both equations, and substitute them into the third Pythagorean trig identity.

3970. The equation of the circle is $x^2 + y^2 = 25$, and the equations of the sides of the central stripe are $x = \pm 1$. Consider only the area in the positive quadrant.

3971. (a) i. Use $a = \frac{v-u}{t}$.

ii. Scale the mass emerging in one second by the scale factor of the time intervals.

(b) Consider $F = ma$ with the expressions found in (a). It can be easier to visualise the tablets emerging in discrete packets of the mass from (a) ii., which are then constantly accelerated, reaching 1.4 ms^{-1} (and hence requiring no more horizontal force) at the instant the next packet lands on the belt.

3972. (a) $a - b < 0$, so, when $|a - b|$ appears, rewrite as $-(a - b)$. You are looking to show that $P_4 = P_2$, which guarantees period 2.

(b) $a - b \geq 0$, so, when $|a - b|$ appears, rewrite as $(a - b)$. Also $a - 2b < 0$, so, when $|a - 2b|$ appears, rewrite as $-(a - 2b)$. You are looking for $P_5 = P_3$, which guarantees period 2.

3973. Approach this graphically. Sketch both equations on the same set of axes. This should establish a point at which they intersect. To show that they don't intersect again, find the gradient of $y = \operatorname{arccot} x$: rearrange to $x = \cot y$. Compare to the gradient of the line.

3974. Show that the function has a turning point in this domain. Look for SPs, and then test the second derivative to guarantee that the function turns.

3975. Take out a common factor of $x^2 - 1$. Factorise what remains as a difference of two squares. Continue algebraically until you find four single roots and a double root. Based on these, sketch a positive sextic.

3976. Convert the words into algebra. The distribution of the sample means is

$$\bar{X} \sim N\left(215, \frac{16^2}{n}\right).$$

You're looking to solve $\mathbb{P}(|\bar{X} - 215| > 10) = 0.01$. Split the probability 0.01 into two tails. So, you need the z-value for $p = 0.005$. Find this with the inverse normal facility on a calculator.

3977. (a) The parabola remains monic. So, you only need to consider the new locations of the roots.

(b) Think of reflection in $y = 0$, followed by a translation by $2q$ in the y direction.

3978. (a) Differentiate $\ln x$ and $\frac{1}{x}$, and use L'Hôpital's rule.
   (b) Integrate by parts (let $u = \ln x$ and $\frac{dv}{dx} = 1$) before taking the limit. You'll need the result from (a).
   (c) Sketch $y = \ln x$, together with the relevant signed areas.

3979. Find $\dot{x}$ and $\dot{y}$. Then set up the equation $\dot{y} = 2\dot{x}$. Find the velocities explicitly having solved: not everything that satisfies the equation is a solution of the original problem.

3980. Set $l = 1$. The base is an equilateral triangle. Find the distance from one of its vertices to its centre: use the fact that the centre of a triangle divides its height in the ratio $1 : 2$, or use Pythagoras. Then set up a vertical right-angled triangle, and use trig. Multiply by l at the end.

3981. Use a counting approach. The possibility space contains $6^3 = 216$ outcomes. Of these, successful outcomes are $(1, 1, 5)$, $(1, 1, 6)$ and $(1, 2, 6)$. Count up the number of orders of each of these, and then use $p = \frac{\text{successful}}{\text{total}}$.

3982. (a) Set the denominator to zero and solve. There is also a horizontal asymptote.
   (b) You don't need to use calculus. A quadratic is symmetrical, hence this function is too. There must be a stationary value halfway between the vertical asymptotes.
   (c) Find first the range of the denominator, then the fraction, then the function.
   (d) Find the roots of the function. In particular, look at the relationship between the x values at the roots, vertical asymptotes, and the SP. Sketch the curve. Consider what will happen, when running N-R, if a starting value x_0 is chosen that is outside the vertical asymptotes.

3983. Consider the following possibilities for the reaction forces at the supports:
   ① $R_{\text{outer}} = \frac{1}{3}mg, \quad R_{\text{inner}} = 0,$
   ② $R_{\text{outer}} = 0, \qquad R_{\text{inner}} = mg.$

3984. Draw a sketch of $y = f(x)$ (any generic cubic will do), and use the rotational symmetry to show that

$$f(a + k) - b = b - f(a - k).$$

Differentiate this with respect to x.

3985. (a) Consider the cases $k < 0$ and $k \geq 0$ separately.
   (b) You can write the larger probability as the smaller plus something positive.

3986. (a) Sub $x = 2$ into the piecewise definitions, and find k such that the values produced match.
   (b) Differentiate each piecewise definition, and verify that the values of $f'(2)$ match.

3987. Two are true, one is false.

3988. Solve the first inequality by considering the signs of the factors. The boundary lines are $x = a$ and $y = b$. The second inequality describes the region outside a circle.

3989. Consider the boundary case, in which the cans are still on the ground, but the reaction force between ground and cans is zero.

3990. No calculation is needed. Consider the symmetry of a cubic.

3991. (a) Rearrange $xe^y = 1$ to make y the subject.
   (b) Use the N-R method on $x - \ln x - 2 = 0$, with starting points $x_0 = 0.1$ and then $x_0 = 2$.
   (c) Set up a definite integral for the y value of the line minus the y value of the curve, with the limits as calculated in (b).

3992. Consider the equation $f(x) = 2x$. Show that it has no roots. Hence, show that $f(x) > 2x$ for all x.

3993. (a) Set up $\frac{\Delta y}{\Delta x}$.
   (b) And again.
   (c) Show that the product of the gradients is -1.

3994. Separate the variables and integrate, including a constant of integration. Exponentiate the whole equation, converting the additive constant into a multiplicative one. Then substitute in the initial conditions.

3995. Assume, for a contradiction, that there are $a, b \in \mathbb{N}$ such that $ax + by = 1$, and also that x and y have a common factor $k > 1$.

3996. (a) Let $y = g(x)$. Solve by separation of variables to find the general solution. Then substitute in $x = 0$, $y = 1$ to find the particular solution.
   (b) Refer back to the original DE.

3997. You can replace **a** and **b** by **i** and **j** wlog.

Equate the coefficients of **a** and solve for t. Then substitute this value of t into the coefficients of **b** and find the coordinates of the point at which the particles collide.

3998. Find the gradient in terms of a. Then substitute this and $(a, a^{\frac{1}{3}})$ into $y - y_1 = m(x - x_1)$. Also substitute $x = -2/3$, $y = 0$ and solve for a.

3999. Sum the geometric series using

$$S_\infty = \frac{a}{1-r}.$$

Considering both a and r as variables which may depend on t, differentiate this (implicitly) by the quotient rule.

4000. (a) The second derivative test is inconclusive. So, consider the behaviour either side of $x = 0$, particularly the symmetry of the graph.

(b) Consider the relative sizes of x^4 and 1.

(c) The graph is even, so has the y axis as a line of symmetry.

——— End of Volume IV ———